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A classical constant of motion with discontinuities

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Abstract. For any central potential, it is possible to construct a constant of the motion which generalises the Runge–Lenz vector. This is a vector pointing from the centre of force to the nearest point of the orbit where r is maximum (or minimum). In general, the direction of this 'constant' vector thus changes abruptly whenever r is minimum (or maximum).

It is well known (Runge 1919, Lenz 1924) that a particle of mass m moving in a central potential $V(r) = -k/r$ has, besides the familiar constants of the motion E and $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, another one:

$$\boldsymbol{M} = (1/m)(\boldsymbol{p} \times \boldsymbol{L}) - k(\boldsymbol{r}/r) \tag{1}$$

which is a vector parallel to the major axis of the ellipse (or hyperbola). One is naturally tempted to generalise this result for an arbitrary $V(r)$. Consider

$$\boldsymbol{M} = a(r)(\boldsymbol{p} \times \boldsymbol{L}) + b(r)\boldsymbol{r} \tag{2}$$

where $a(r)$ and $b(r)$ are functions to be determined, which may also depend on the constants E and L^2. To compute $\dot{\boldsymbol{M}}$, we use $\dot{\boldsymbol{p}} = -\boldsymbol{r}V'(r)/r$ and $\dot{\boldsymbol{r}} = \boldsymbol{p}/m$. In the resulting equation, it is convenient to write $\boldsymbol{p} \times \boldsymbol{L} = p^2\boldsymbol{r} - (\boldsymbol{r} \cdot \boldsymbol{p})\boldsymbol{p}$, so as to make all the terms of $\dot{\boldsymbol{M}}$ proportional to $\boldsymbol{r}$ or to $\boldsymbol{p}$. Setting $\dot{\boldsymbol{M}} = 0$, we obtain as the coefficient of $\boldsymbol{r}(\boldsymbol{r} \cdot \boldsymbol{p})/r$

$$a'p^2 - maV' + b' = 0 \tag{3}$$

and as the coefficient of $\boldsymbol{p}$

$$-a'(\boldsymbol{r} \cdot \boldsymbol{p})^2/r + maV'r + b = 0. \tag{4}$$

In these equations, we can substitute $(\boldsymbol{r} \cdot \boldsymbol{p})^2 = r^2p^2 - L^2$ and $p^2 = 2m(E - V)$. Since E and L^2 are constants of the motion, we have two differential equations for $a(r)$ and $b(r)$. It is convenient to multiply equation (3) by r and add the result to (4), thus obtaining

$$(br)' + a'L^2/r = 0 \tag{5}$$

which is independent of the potential. It is also convenient to introduce the notation $Z(r) = m(E - V) = p^2/2$, so that (3) becomes

$$2a'Z + aZ' + b' = 0. \tag{6}$$

Alternatively, we can eliminate b and obtain a second-order equation for a, namely

$$a''(2rZ - L^2/r) + a'(4Z + 3rZ' + L^2/r^2) + a(2Z' + rZ'') = 0. \tag{7}$$

As Z is a known function of r, this equation can be continuously integrated unless the

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coefficient of a'' vanishes. The latter is

$$(r^2p^2 - L^2)/r = (\boldsymbol{r}\,.\,\boldsymbol{p})^2/r = m^2 r(\dot{r})^2. \tag{8}$$

The difficulty now becomes apparent: equation (7) is singular at the turning points of the orbit! It is still possible to choose a solution which is regular at one of them, but not at both (with some exceptions, such as Kepler orbits where the perihelion, the centre of force and the aphelion are collinear).

The situation is illustrated in figure 1. From

$$\boldsymbol{M} = (2aZ + b)\boldsymbol{r} - a\boldsymbol{p}(\boldsymbol{p}\,.\,\boldsymbol{r}) \tag{9}$$

it is obvious that when $\boldsymbol{p}\,.\,\boldsymbol{r} = 0$, $\boldsymbol{M}$ is parallel to $\boldsymbol{r}$, *provided that a is finite.* In figure 1, we have chosen a solution where $a(r_{\mathrm{max}})$ is finite, but $a(r_{\mathrm{min}})$ is not. Thus $\boldsymbol{M}$ points along the line connecting the centre of force with the points A, B, C, D . . . and changes its direction abruptly when r is minimum.

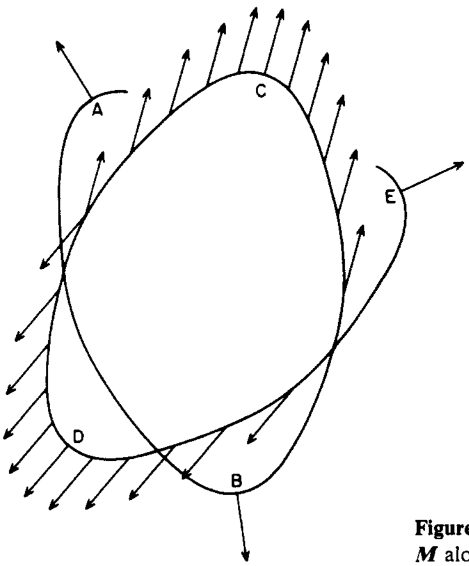

Figure 1. Behaviour of the 'constant' vector $\boldsymbol{M}$ along the orbit.

As an example, consider the truncated Kepler potential

$$\begin{aligned} V &= -k/r \qquad && r > r_0 \\ V &= -k/r_0 \qquad && r < r_0. \end{aligned} \tag{10}$$

A bound orbit consists of elliptic arcs (for $r > r_0$) connected by straight segments (for $r < r_0$). For $r > r_0$, the regular solution is given by equation (1):

$$a = m^{-1} \qquad b = -k/r.$$

For $r < r_0$, we have from equation (6)

$$a' = -b'/2Z_0 \tag{11}$$

where $Z_0 = m(E + k/r_0)$ is the constant value of $Z(r)$ when $r < r_0$. Substitution into equation (5) then yields

$$b = -(k/r_0)[(r_0^2 - L^2/2Z_0)/(r^2 - L^2/2Z_0)]^{1/2} \tag{12}$$

where the constant of integration has been adjusted so that b is continuous at $r = r_0$. Likewise

$$a = m^{-1} - (b + k/r_0)/2Z_0 \tag{13}$$

is continuous there.

We now see how a and b become singular when $r^2 = L^2/2Z_0$ that is when r is minimum. Since $2Z_0 r^2 - L^2 = p^2 r^2 - L^2 = (\boldsymbol{p} \cdot \boldsymbol{r})^2$, their singularity behaves as $1/|\boldsymbol{p} \cdot \boldsymbol{r}|$. The last term in equation (9) then behaves as

$$\boldsymbol{p} \cdot \boldsymbol{r}/|\boldsymbol{p} \cdot \boldsymbol{r}| = \dot{r}/|\dot{r}| \tag{14}$$

which is finite but discontinuous when r is minimum.

On the other hand, the first term of equation (9) is continuous because $2aZ + b$ is constant for $r < r_0$, by virtue of equation (11).

This example shows explicitly how the vector $\boldsymbol{M}$ remains constant for finite lapses of time and changes its direction abruptly when r is minimum.

References

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